Constructing $1/\omega^{\alpha}$ noise from reversible Markov chains

Sveinung Erland*
Department of Mathematics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

Priscilla E. Greenwood
Department of Mathematics and Statistics, Arizona State University, Tempe, Arizona 85287, USA

(Received 15 February 2007; published 12 September 2007)

This paper gives sufficient conditions for the output of $1/\omega^{\alpha}$ noise from reversible Markov chains on finite state spaces. We construct several examples exhibiting this behavior in a specified range of frequencies. We apply simple representations of the covariance function and the spectral density in terms of the eigendecomposition of the probability transition matrix. The results extend to hidden Markov chains. We generalize the results for aggregations of AR1-processes of C. W. J. Granger [J. Econometrics **14**, 227 (1980)]. Given the eigenvalue function, there is a variety of ways to assign values to the states such that the $1/\omega^{\alpha}$ condition is satisfied. We show that a random walk on a certain state space is complementary to the point process model of $1/\omega$ noise of B. Kaulakys and T. Meskauskas [Phys. Rev. E **58**, 7013 (1998)]. Passing to a continuous state space, we construct $1/\omega^{\alpha}$ noise which also has a long memory.

DOI: 10.1103/PhysRevE.76.031114 PACS number(s): 05.40.−a, 02.50.−r, 72.70.+m

I. INTRODUCTION

A real-valued stationary time series $\{g(X_t),\ t\in\mathbb{Z}\}$ is said to produce $1/\omega^{\alpha}$ noise if the spectral density satisfies

$$f(\omega)\sim\frac{1}{\omega^{\alpha}},\quad L\ll\omega\ll H, \tag{1}$$

where $(L,H)\subset[0,\pi]$ is a relatively long interval of frequencies, α is positive and close to 1, and $\sim$ and $\ll$ denote certain asymptotic behaviors, more precisely defined in Sec. III. The phenomenon has been observed in a wide variety of fields including astrophysics, electrical engineering, geophysics, hydrology, economics, acoustics, medicine, and psychology, with α usually being in the interval (0.5, 1.5) [1–3]. Even the sound intensity of classical music shows this behavior over several decades of frequencies.

The challenge posed for stochastic modelers, then, is to find a general construction which results in $1/\omega^{\alpha}$ noise, and explain it arising in so many situations. Many of these phenomena can reasonably be viewed as either discrete or continuous time processes which evolve with memory of only a limited past. It seems reasonable to think in terms of Markov chains or Markov processes.

In this paper we formulate, in terms of their eigenstructure, a general construction of reversible Markov chains which produce $1/\omega^{\alpha}$ noise. The construction provides a basis for understanding some instances of this phenomenon. As illustration we construct families of random walks, of renewal processes, and of Markov chain Monte Carlo (MCMC) samplers which produce $1/\omega^{\alpha}$ noise. We expect that the construction is robust in the sense that families which loosely follow the construction will have nearly $1/\omega^{\alpha}$ behavior on a large frequency interval. While not a final answer to the challenge of explaining the frequent appearance of $1/\omega^{\alpha}$ noise, our construction provides a basis for understanding that this phenomenon can arise in a variety of situations from common structural roots.

Among probabilists and statisticians the main focus related to the phenomenon described by (1) has been on the construction of so-called long-memory processes [1,4,5], characterized by

$$f(\omega)\sim\frac{1}{\omega^{\alpha}}\quad\text{as}\quad\omega\to 0. \tag{2}$$

This behavior is, however, neither sufficient nor necessary for relation (1). Notice that observations and inference regarding the spectral density are limited to a finite interval of frequencies bounded away from zero, where the sampling frequency constitutes the upper bound, and the length of the observed time series determines the lower bound; and even though $f(\omega)$ complies with (2), it may differ significantly from $1/\omega^{\alpha}$ on the interval of interest, which does not include zero. On the other hand, unless L is zero, relation (1) does not force any particular behavior for ω in a neighborhood of zero. Values of $\alpha\geq 1$ are usually not considered for models satisfying (2) since these imply nonstationarity [6,7]. Empirical findings where $f(\omega)$ complies with (1), but not with (2) [3] suggest that the focus on (2) may lead attention away from the actual problem of interest. In this paper we will focus on stationary Markov chains exhibiting the characteristic in (1) for $\alpha\in(0,2)$.

Fractional Brownian motion and fractional Gaussian noise [8] are well-studied models having spectral density proportional to $1/\omega^{\alpha}$ over all frequencies. Wavelet-based processes can produce $1/\omega^{\alpha}$ noise up to any accuracy [9]. This is also the case for the aggregation of relaxation processes (e.g., first-order autoregressive (AR1) processes) with an appropriate distribution of time constants. The latter is perhaps the most frequently proposed explanation for $1/\omega^{\alpha}$ noise [4,10]. The spectral density of the voltage over an electrical circuit with certain configurations of resistors and capacitors can be

*Present address: Gassco AS, P.O. Box 93, N-5501 Haugesund, Norway.

explained in this way [3,6]. In the next section, we will show that a Markov chain may have the same type of spectral density as an aggregation of relaxation processes.

In this paper we construct processes, which produce $1/\omega^{\alpha}$ noise on an arbitrarily long interval in log scale, bounded away from zero. There seem to be few counterparts to this in the literature. One other example, is a point process where the interevent times are given by a Gaussian AR1 time series and $\alpha=1$ [11].

In the physics literature, there are a number of more complex models for which property (1) is claimed [2,12–16]. One sees that the characteristic in (1) is often given the generic term $1/f$ noise. Surveys of $1/f$ phenomena and ideas are found in [6,7,10], with some recent results in [17–19].

Let $\mathcal{X}$ be the state space of X_t, and let $g:\mathcal{X}\to\mathbb{R}$ be a function that assigns real values to the abstract state space. The one-sided spectral density of a stationary time series $\{g(X_t)\}$ is defined as

$$f(\omega) \equiv \lim_{N\to\infty}\frac{1}{N\pi}\left|\sum_{t=1}^{N} g(X_t)e^{-i\omega t}\right|^2 \tag{3}$$

$$=\frac{1}{\pi}\sum_{t=-\infty}^{\infty} C(t)e^{-i\omega t}, \tag{4}$$

where $\omega\in[0,\pi]$ and $C(t)=\mathrm{Cov}[g(X_s),g(X_{s+t})]$ is the stationary covariance function. The equivalence of (3) and (4) is the Wiener-Khinchin theorem. For nonstationary time series, an extended definition in [6] can be used. The partial sum of $\{g(X_t)\}$ does not always strictly possess a spectrum, but in an extended sense it will have the same spectral density as $\{g(X_t)\}$ divided by $2[1-\cos(\omega)]=\omega^2+\mathcal{O}(\omega^4)$ [20]. In order to construct $1/\omega^{\alpha}$ noise for $\alpha>0$ and $\omega<1$, it is therefore sufficient to consider $\alpha\in(0,2]$.

The problem of finding an analytical representation of the spectral density is one of the main difficulties in the analysis of $1/\omega^{\alpha}$ noise. An Abelian-Tauberian theorem relating regularly varying tails shows that the long-range dependence property (2) is equivalent to similar behavior of $C(t)$ as $t\to\infty$ [21]. But for (1), the entire function $C(t)$ has to be taken into consideration. Often $C(t)$ is hard to compute. Some results have emerged in the setting of point processes, which are generally easier to handle by (3) (see, e.g., [11]). Here we present a simple expression for $C(t)$ for a Markov chain on a finite abstract state space in terms of the eigenvalues and eigenvectors of the chain and a map which assigns real values to the abstract states. For a sequence of Markov chains with increasing state spaces we find conditions on the sequence of eigenstructures and maps such that the spectra of the chains approach $1/\omega^{\alpha}$ as exemplified in Sec. IV.

This paper is organized as follows: In Sec. II the covariance function and spectral density are computed using an eigendecomposition. A sufficient condition for limiting $1/\omega^{\alpha}$ behavior is presented in Sec. III and used to generalize the results of Granger [4]. In Sec. IV we construct some specific chains satisfying this condition. A random walk in a nonhomogeneous environment and a hidden Markov chain satisfying (1) are constructed by assigning certain nonmonotonic values to the states. The number of ways to do this increases geometrically with the number of states. Other types of random walk are also shown to exhibit certain $1/\omega^{\alpha}$ behaviors. A Metropolized independence sampler evolving on equidistant, ordered numbers can be extended to satisfy both (1) and (2).

II. COVARIANCE FUNCTION AND SPECTRAL DENSITY

A. Eigendecomposition of a probability transition matrix

Let P be an aperiodic, irreducible $m\times m$ transition probability matrix with linearly independent eigenvectors. Let the m-vector $\boldsymbol{\pi}$ be the stationary distribution of P, and let P^T be its conjugate transpose. Then for $t=0,1,2,\ldots$ [22],

$$P^t = E\Lambda^t F^T = \Pi + \sum_{k=1}^{m-1}\lambda_k^t \boldsymbol{e}_k\boldsymbol{f}_k^T, \tag{5}$$

where $\{\boldsymbol{e}_k\}$ and $\{\boldsymbol{f}_k\}$ are the right and left eigenvectors of P and the columns in E and F, respectively. The eigenvectors are normalized such that $F^TE=I$. The diagonal matrix $\Lambda=\mathrm{diag}(\lambda_k,\ k=0,\ldots,m-1)$ consists of the corresponding eigenvalues. The matrix $\Pi=\lim_{t\to\infty}P^t$ has the stationary distribution $\boldsymbol{\pi}^T$ on each row, since $\lambda_0=1$, $\boldsymbol{e}_0=[1,\ldots,1]^T$, and $\boldsymbol{f}_0=\boldsymbol{\pi}$. The chain is aperiodic if and only if $\min_k\lambda_k>-1$.

Let the matrix $B\equiv\mathrm{diag}(\boldsymbol{\pi})$. That P is time reversible ($BP=P^TB$) is equivalent with $F=BE$ and all λ_k being real-valued. This simplifies the analytical treatment of the spectral density, and in this paper we therefore only consider reversible chains. This does, however, not exclude $1/\omega^{\alpha}$-behavior of nonreversible Markov chains [23]. Eigenanalysis of reversible Markov chains is done in, e.g., [24–26]. We suppress the dependence on m in the notation until Sec. III.

B. Covariance function and spectral density

For analyzing the spectral density of a reversible Markov chain on a finite state space, we use the expression for the covariance function (6), given in [27,28]. From this we derive the spectral density (7), which also appears in [29]. We regard the initial state space as abstract.

Lemma 1. Assume that an aperiodic, irreducible Markov chain $\{X_t\}$ has a probability transition matrix P with right and left eigenvectors $\boldsymbol{e}_k$ and $\boldsymbol{f}_k$, respectively, and eigenvalues λ_k, $k=0,\ldots,m-1$, where m is the number of states in the state space. Let $\boldsymbol{g}$ be any vector assigning real values to the states. Then

1. $\{g(X_t)\}$ has the covariance function

$$C(t) = \sum_{k=1}^{m-1} a_k\lambda_k^{|t|}. \tag{6}$$

2. $\{g(X_t)\}$ has the spectral density

$$f(\omega) = \frac{1}{\pi}\sum_{k=1}^{m-1} \frac{a_k(1-\lambda_k^2)}{1+\lambda_k^2 - 2\lambda_k \cos(\omega)}, \tag{7}$$

where $a_k \equiv (\boldsymbol{g}^T B\boldsymbol{e}_k)(\boldsymbol{f}_k^T\boldsymbol{g})$. The matrix B is related to the chain as above.

Similar results for Markov processes on finite state spaces are presented in [28] and for Markov chains on continuous state spaces in [30]. Note that when all the eigenvalues λ_k are real, each term in (6) is equal to the covariance function of an AR1-process with rate λ_k and stationary variance a_k. Each term in (7) is the spectral density of the corresponding AR1 process (see Sec. III).

In Sec. III we show that it is the relation between $\{a_k\}$ and $\{\lambda_k\}$ which determines whether a Markov chain produces $1/\omega^\alpha$ noise. The chain $\{X_t\}$ evolves on an abstract state space, and we may choose $\boldsymbol{g}$ to be a map to the ordered natural numbers. There is a variety of ways to choose $\boldsymbol{g}$, which yields a particular sequence $\{a_k\}_{k=1}^{m-1}$ in (6) and (7). We can, e.g., construct $\boldsymbol{g}$ by taking a sequence $\{b_k\}_{k=0}^{m-1}$ of real numbers and then set

$$\boldsymbol{g} = \sum_{k=0}^{m-1} b_k \boldsymbol{e}_k. \tag{8}$$

Then $a_k = |\boldsymbol{g}^T B\boldsymbol{e}_k|^2 = b_k^2$ if the Markov chain is reversible. Hence there are more than 2^{m-1} ways to choose $\{b_k\}_{k=0}^{m-1}$ in order to obtain a desired sequence $\{a_k\}_{k=1}^{m-1}$, since the sign of b_k does not matter, and b_0 can be chosen freely.

C. Hidden Markov chains

A hidden Markov chain is a pair of processes $\{X_t, Y_t\}$, where $\{X_t\}$ is an unobserved Markov chain and $\{Y_t\}$ is a Markov chain conditionally on $\{X_t\}$. For us the observed chain is $\{\tilde{g}(Y_t)\}$, where the state space $\tilde{\mathcal{X}}$ of Y_t consists of $n \geq 2$ different states and the function $\tilde{g}: \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ assigns real values to $\tilde{\mathcal{X}}$. Here we restrict to the case where, given X_t, Y_t is generated by an $m \times n$ probability transition matrix Q, such that all Y_t are independent given $\{X_t\}$. We have the following result:

Lemma 2. Let $\{\tilde{g}(Y)_t\}$ denote the observed values of a hidden version of the Markov chain $\{X_t\}$ in Lemma 1. Let Q be an $m \times n$ probability transition matrix such that

$$Q(x,y) = \Pr(\,Y_t = y | X_t = x).$$

Further let $\tilde{\boldsymbol{g}}$ be an n vector which assigns values to all the states in the state space of Y_t. Then the covariance function and spectral density of $\{\tilde{g}(Y_t)\}$ are given by (6) and (7), respectively, with

$$a_k \equiv (\tilde{\boldsymbol{g}}^T Q^T B\boldsymbol{e}_k)(\boldsymbol{f}_k^T Q\tilde{\boldsymbol{g}}).$$

Notice that when X_t is a reversible Markov chain, $a_k = |\boldsymbol{e}_k^T BQ\tilde{\boldsymbol{g}}|^2$ for the hidden chain.

III. SUFFICIENT CONDITION FOR $1/\omega^\alpha$ NOISE

In this section we will show for a sequence of Markov chains $\{X_t^m\}$ that if there is an appropriate relationship between $\{a_{k,m}\}$ and $\{\lambda_{k,m}\}$, then the sequence will produce $1/\omega^\alpha$ noise, that is, the spectral densities are asymptotically proportional to $1/\omega^\alpha$ on long intervals in log scale. We first define exactly what we mean by these words.

Definition 1. Let $\{h_m(\omega)\}$ be a sequence of continuous functions and $\{(L_m, H_m)\}$ an associated sequence of intervals. Let $\phi(\omega)$ be a continuous function. We say $h_m(\omega)$ is asymptotically proportional in log-scale to $\phi(\omega)$ on the interval (L_m, H_m), and write

$$h_m(\omega) \sim \phi(\omega) \quad \text{for} \quad L_m \ll \omega \ll H_m,$$

if when $L_m/H_m \rightarrow 0$, there exists a sequence $\{c_m\}$ of positive real numbers such that

$$\lim_{m\rightarrow\infty} \left| \frac{c_m h_m(\omega_m)}{\phi(\omega_m)} - 1 \right| = 0$$

for any sequence $\{\omega_m\}$, which satisfies $L_m/\omega_m \rightarrow 0$ and $H_m/\omega_m \rightarrow \infty$.

If, e.g., $L_m = 1/m^2$ and $H_m = 1/m$, then $c_m h_m(\omega_m)$ can get arbitrarily close to $\phi(\omega_m)$ both for $\{\omega_m\}$ converging to zero almost as fast as $\{L_m\}$, and for $\{\omega_m\}$ converging almost as slow as $\{H_m\}$. This is also true for those $\{\omega_m\}$ in between, resulting in an arbitrarily long interval in log-scale, contained in the interval $[\log(L_m), \log(H_m)]$.

The common definition of asymptotic behavior as $\omega \rightarrow 0$ is that $h(\omega)/\phi(\omega) \rightarrow 1$, written as (2) in [31]. Definition 1 extends to this definition when $L_m \equiv 0$, $H_m \equiv 1$, and $c_m h_m(\omega) \equiv h(\omega)$. Notice that when $L_m > 0$, the functions $h_m(\omega)$ are not necessarily asymptotic to $\phi(\omega)$ as $m \rightarrow \infty$ and $\omega \rightarrow 0$, but may have this asymptotic property in m only on an ω interval bounded away from zero. This is an important distinction in view of the fact that total power must be finite in examples, and other issues pointed out in the Introduction. We allow the proportionality constants c_m because we are interested in the shape of $h_m(\omega)$, not its level.

As an example where Definition 1 holds, consider a sequence of AR1-processes, given by

$$X_t^m = \beta_m X_{t-1}^m + \varepsilon_t, \tag{9}$$

where all ε_t are independent and identically distributed with $E[\varepsilon_t] = 0$, $\mathrm{Var}[\varepsilon_t] = \sigma^2$, and where $\{\beta_m\}$ is a sequence of numbers in (0, 1). The spectral densities of a sequence of AR1-processes have the properties

$$f_m^{\mathrm{AR1}}(\omega) = \frac{\sigma^2}{(1-\beta_m)^2 + 2\beta_m[1-\cos(\omega)]} \sim \begin{cases} 1, & 0 \ll \omega \ll 1-\beta_m \\ \dfrac{1}{\omega^2}, & 1-\beta_m \ll \omega \ll 1. \end{cases} \tag{10}$$

The conditions of Definition 1 will be fulfilled in the first statement in (10) with $\phi(\omega) = \omega^0 = 1$ if $\omega_m/(1-\beta_m) \rightarrow 0$, and if we choose $c_m = \sigma^2/(1-\beta_m)^2$. The conditions of Definition 1 will be fulfilled in the second statement in (10) with $\phi(\omega) = 1/\omega^2$ if $(1-\beta_m)/\omega_m \rightarrow 0$ and $\omega_m \rightarrow 0$, and if we choose $c_m = \sigma^2$.

Definition 2. A sequence of time series is said to produce

$1/\omega^\alpha$ noise for $L_m \ll \omega \ll H_m$, if the spectral densities $f_m(\omega) \sim 1/\omega^\alpha$ for $L_m \ll \omega \ll H_m$. When we speak of constructing a $1/\omega^\alpha$ process, or of a Markov chain producing a $1/\omega^\alpha$ noise, we mean that we construct such a sequence of time series.

A. Constructing $1/\omega^\alpha$ finite state Markov chains

The relationship between $\{a_{k,m}\}$ and $\{\lambda_{k,m}\}$ which yields $1/\omega^\alpha$ noise can be deduced heuristically by setting $a_{k,m}$ so that each term in the sum in the covariance function equals the function $1/\omega^\alpha$ at $\omega=(1-\lambda_{k,m})$. At this point the term changes from $1/\omega^0$ behavior to $1/\omega^2$ behavior as in (10). We get that

$$\frac{a_{k,m}(1-\lambda_{k,m}^2)}{(1-\lambda_{k,m})^2+2\lambda_{k,m}[1-\cos(1-\lambda_{k,m})]} = \frac{1}{(1-\lambda_{k,m})^\alpha},$$

which, for $\lambda_{k,m}$ close to 1, implies that

$$a_{k,m} \approx (1-\lambda_{k,m})^{1-\alpha}. \qquad (11)$$

If one plots the spectral density of a sum of AR1-processes using the rule-of-thumb in (11), one observes $1/\omega^\alpha$ behavior over several decades in log-log-scale for relatively small m ($m \geq 3$). For $m \to \infty$, we may combine (11) with the assumption that $\log(1-\lambda_{k,m})$ is equidistant and treat the spectral density of the Markov chain as a Riemann-sum. We then obtain the following result.

Theorem 1. Let X_t^m be a sequence of Markov chains as in Lemma 1. Let $\boldsymbol{g}$ be a vector assigning real values to the states, and let $\{a_{k,m}\}$ be the corresponding coefficients appearing in Lemma 1.

Let $L_m \equiv 1-\lambda_{1,m}$, $H_m \equiv 1-\lambda_{m-1,m}$, and $\alpha \in (0,2)$. Assume there exist continuous functions $a(x)$ and $\lambda(x)$ for $x \in (0,1)$, and a sequence $\{A_m\}$, such that $a_{k,m}=A_m a(k/m)$ and $\lambda_{k,m}=\lambda(k/m)$. Assume that $\lambda(x)$ is invertible and analytic at $x=0$, and that $\lambda(0)=1$.

Assume that there exists a constant C, such that

$$\lim_{x\to 0} \frac{a(x)[1-\lambda(x)]^\alpha}{|\lambda'(x)|} = C, \qquad (12)$$

and assume that $a(x)[1+\lambda(x)][1-\lambda(x)]^\alpha/|\lambda'(x)|$ is bounded for $x \in (0,1)$.

Then the spectral densities, given by (7), satisfy

$$f_m(\omega) \sim \begin{cases} 1, & \text{for } 0 \ll \omega \ll L_m \\ \dfrac{1}{\omega^\alpha}, & \text{for } L_m \ll \omega \ll H_m \\ \dfrac{1}{\omega^2}, & \text{for } H_m \ll \omega \ll 1. \end{cases}$$

In order to obtain examples where Theorem 1 holds, the main challenge is to satisfy (12). If, for instance,

$$a(x) = \frac{|\lambda'(x)|}{[1-\lambda(x)]^\alpha}, \qquad (13)$$

the condition in (12) is satisfied. Notice that we allow negative eigenvalues. However, the terms in (7) with negative eigenvalues are increasing in ω, and this might imply that the $1/\omega^\alpha$ behavior can be observed only for very small ω.

In Sec. IV we will use these results to construct a variety of Markov chains producing $1/\omega^\alpha$ noise.

B. Aggregation of AR1-processes

Theorem 1 can be used to generalize a result of Granger for aggregations of independent AR1-processes [4]. For convenience we let the coefficients $\beta \in (0,1)$ in the AR1-processes in (9) be independent with density $h(\beta)$. Consider the sequence of processes formed by aggregation of many such AR1-processes with coefficients β_i, and independent ε_t's with equal variances,

$$Y_t^n = \frac{1}{n}\sum_{i=1}^{n} X_t^i. \qquad (14)$$

Granger let the coefficients β_i follow a beta-distribution on (0, 1). He indicated that the shape of the distribution of the β_i is important for his result only near $\beta=1$. His result is equivalent to the limiting spectral density, as n increases, being asymptotic to $1/\omega^\alpha$ as $\omega \to 0$ for $\alpha \in (0,1)$. The following Corollary to Theorem 1 shows a sufficient condition on h for this property to hold.

Corollary 1. Consider the sequence of aggregated AR1-processes in (14). Let $h(\beta)$ be the probability density of the coefficients, and let $\alpha \in (0,1)$. If there exists a constant C such that

$$\lim_{\beta\to 1} \frac{h(\beta)}{(1-\beta)^{1-\alpha}} = C, \qquad (15)$$

then the limiting spectral density $f(\omega) \sim 1/\omega^\alpha$ for $\omega \to 0$.

It is straightforward to verify that the parameters in the beta distribution can be chosen to satisfy (15). However, if $\alpha \geq 1$, the resulting process has infinite variance, and the spectral density is not defined.

A version of Theorem 1, where the asymptotic behavior holds on intervals which may be bounded away from 0, also holds for this example. Note that this result is distinct from the previous one, neither being an extension of the other.

Corollary 2. Consider the sequence of aggregated AR1-processes in (14). Let $\alpha \in (0,2)$, and let the function $h(\beta)$ satisfy (15). Assume that the probability density of the coefficients at stage n is proportional to h on the support $(\beta_n^{\min}, \beta_n^{\max}) \subset (0,1)$. Then the spectral densities

$$f_n(\omega) \sim \begin{cases} 1, & \text{for } 0 \ll \omega \ll 1-\beta_n^{\max} \\ \dfrac{1}{\omega^\alpha}, & \text{for } 1-\beta_n^{\max} \ll \omega \ll 1-\beta_n^{\min} \\ \dfrac{1}{\omega^2}, & \text{for } 1-\beta_n^{\min} \ll \omega \ll 1. \end{cases}$$

Note that the probability density of the coefficients in Corollary 2 depends on n.

IV. CONSTRUCTIONS OF $1/\omega^{\alpha}$ NOISE

In this section we will present several constructions of $1/\omega^{\alpha}$ processes. In Sec. IV A this is done for $\alpha\in(0,2)$ by designing the state space $\boldsymbol{g}$ according to the eigenvalue function of a given transition probability matrix. The chain is characterized by having trajectories with many relatively small steps and intermittent larger steps coming in bursts, which is common for many $1/\omega^{\alpha}$ phenomena. In Sec. IV B we show that a random walk on a certain state space is complementary to the point process model of $1/\omega$ noise of B. Kaulakys and T. Meskauskas [11]. In Sec. IV C we investigate a finite state version of the return times to a central state of a random walk ($\alpha=0.5$), and in Sec. IV D we show how the boundary conditions of a random walk influence the spectrum ($\alpha=1.5$). In Sec. IV E we construct a Metropolized independence sampler, which produces $1/\omega^{\alpha}$ noise [$\alpha\in(0,2)$] while evolving on an ordinary state space of equidistant ordered natural numbers. We also construct the limit process on a continuous state space. All our examples are reversible Markov chains.

A. Random walk in a nonhomogeneous environment

In this section we will construct $1/\omega^{\alpha}$ noise by considering reversible Markov chains with known eigendecompositions. We satisfy the main condition in Theorem 1 by assigning values to the state space according to (8).

Brownian motion has spectrum $1/\omega^2$, and certain random walks exhibit similar behavior. Both exact and approximate eigendecompositions of some random walks on finite state spaces are available. Spitzer [32] presents explicit, exact, eigenvalue analysis for a class of random walks with absorbing boundaries, and results are also available for analogous matrices in two and three dimensions [33].

Feller [22] obtains exact eigeninformation for general cyclical matrices and a class of nearest-neighbor random walks where the probabilities of moving to each of the neighbors can be different. We treat one special case from each of these two classes, and they are called symmetric random walks with reflecting and cyclical boundaries, respectively. Let the respective $m\times m$ transition probability matrices be defined by

$$P_{\mathrm{RW}}=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0\\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0\\ \vdots & \vdots & & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2}\\ 0 & 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2}\end{bmatrix}, \tag{16}$$

$$P_{\mathrm{Cyc}}=\begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2}\\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0\\ \vdots & \vdots & & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2}\\ \frac{1}{2} & 0 & 0 & \cdots & \frac{1}{2} & 0\end{bmatrix}.$$

Lemma 3. The eigenvalues, eigenvectors, and stationary laws of the transition probability matrices in (16), and the coefficients in Lemma 1 for $\boldsymbol{g}=[1,2,\ldots,m]^T$ are given by

$$\lambda_{k,m}^{\mathrm{RW}}=\cos\left(\frac{k\pi}{m}\right),\quad \lambda_{k,m}^{\mathrm{Cyc}}=\cos\left(\frac{2\pi k}{m}\right),$$

$$e_{k,m}^{\mathrm{RW}}(x)=\sqrt{2}\cos\left(\frac{k\pi}{m}(x-0.5)\right),\ e_{k,m}^{\mathrm{Cyc}}(x)=e^{i(2\pi kx/1)}, \tag{17}$$

$$\pi_m^{\mathrm{RW}}(x)=\frac{1}{m},\quad \pi_m^{\mathrm{Cyc}}(x)=\frac{1}{m},$$

$$a_{k,m}^{\mathrm{RW}}=\begin{cases}\dfrac{\cos^2(\pi k/2m)}{2m^2\sin^4(\pi k/2m)}, & k\ \text{odd}\\ 0 & k\ \text{even}\end{cases}\qquad a_{k,m}^{\mathrm{Cyc}}=\frac{1}{4\sin^2\left(\frac{\pi k}{m}\right)} \tag{18}$$

for $k=1,\ldots,m-1$ and $x=1,\ldots,m$.

The eigeninformation for simple random walks with reflecting elastic boundaries and approximate eigeninformation for random walks with discretized Gaussian increments are presented in [34].

The eigenvalues of P_{RW} can be represented as values of a continuous function, $\lambda_{k,m}^{\mathrm{RW}}=\lambda^{\mathrm{RW}}(k/m)$, where

$$\lambda^{\mathrm{RW}}(x)=\cos(\pi x). \tag{19}$$

In order to obtain $1/\omega^{\alpha}$ noise, it is sufficient that $a(x)$ in Theorem 1 satisfies, e.g., (13), which for $\lambda(x)\equiv\lambda^{\mathrm{RW}}(x)$ becomes

$$a(x)=\frac{\pi\sin(\pi x)}{[1-\cos(\pi x)]^{\alpha}}, \tag{20}$$

with coefficients $a_{k,m}=a(k/m)=\mathcal{O}[(k/m)^{1-2\alpha}]$, $k=1,\ldots,m-1$.

If we were to choose $\boldsymbol{g}=[1,2,\ldots,m]^T$, the coefficients $a_{k,m}^{\mathrm{RW}}$ in (18) would be $\mathcal{O}[(k/m)^{-4}]$ for odd k. This is a much faster decrease than prescribed by (20), and the first term in the spectral density in (7) dominates. Hence $f_m(\omega)$ with this choice of $\boldsymbol{g}$ behaves approximately like the spectral density of the AR1-process in (10) with $\beta_m=\lambda_{1,m}^{\mathrm{RW}}$ and $\alpha=2$. This is not unexpected, since our random walks with far-off reflecting boundaries have a strong similarity to Brownian motion.

Nevertheless, by assigning appropriate values to the state space, the desired $a_{k,m}$ sequence, given from (20), can be obtained:

Proposition 1. Consider a sequence of random walks defined by the transition probability matrix P_{RW} in (15) with the eigeninformation $\{\lambda_{k,m}^{\mathrm{RW}}\}$ and $\{\boldsymbol{e}_{k,m}^{\mathrm{RW}}\}$ given in Lemma 3. Let the values assigned to the state space be given by

$$\boldsymbol{g} = \sum_{k=0}^{m-1} b_{k,m}\boldsymbol{e}_{k,m}^{\mathrm{RW}},$$

where $b_{k,m}$ is chosen such that $b_{k,m}^2 = a(k/m)$, $k=1,\ldots,m-1$, and $b_{0,m}$ can be chosen freely. $a(x)$ is given in (20) for some $\alpha \in (0,2)$. Then $a_{k,m}=a(k/m)$, and the sequence of chains produces $1/\omega^{\alpha}$ noise for $(1-\lambda_{1,m}^{\mathrm{RW}}) \ll \omega \ll 1$.

The $b_{k,m}$ are determined only up to sign, so many choices, 2^m of them, are possible. The spectral density and a typical sample path are presented in Fig. 1 for one such choice of $\boldsymbol{g}$, with $b_{k,m}=\sqrt{|a(k/m)|}$, $\alpha=1$, and $a(x)$ given in (20). For this choice

$$g(x) = \sqrt{2\pi}\sum_{k=1}^{m-1} \frac{\sin(\pi k/m)}{1-\cos(\pi k/m)}\cos\left(\frac{k\pi}{m}(x-0.5)\right), \quad (21)$$

for $x=1,\ldots,m$.

There are at least two possible interpretations of the structure of $\boldsymbol{g}$. First the random walk may be simple, but some nonhomogeneous environment is observed instead of the position. Another view is that it is the position that is measured, but the step-size depends on the current state. This observation seems to be relevant for the discussion in [15] where it is suggested that a random walk in a random environment might produce $1/\omega^{\alpha}$ noise.

Notice that one can write conditions corresponding to (20) for P_{Cyc} and for the discretized Gaussian random walks described in [34].

B. Correspondence with point process model of $1/\omega$ noise

In the point process model of $1/\omega$ noise of Kaulakys and Meskauskas the interevent times are given by a Gaussian AR1 time series with very long relaxation times [11,17]. At each event the process takes the same value.

Heuristically, a corresponding time series could arise in a model where the interevent times are equal and the process takes values equal to $X_t = 1/\sqrt{Y_t}$ where Y_t is an AR1 time series corresponding to the one in [11]. In this way, long interevents times are converted to small process values. The power spectral density is derived from the covariance function, which is a second order functional of the signal values. The square root is introduced so that the units in which power is measured are the same as for the point process.

A $1/\omega$ model according to these heuristics can be constructed by letting the underlying process, Y_t, be a random walk with reflecting boundaries, with the eigendecomposition described in Sec. IV A. This random walk behaves

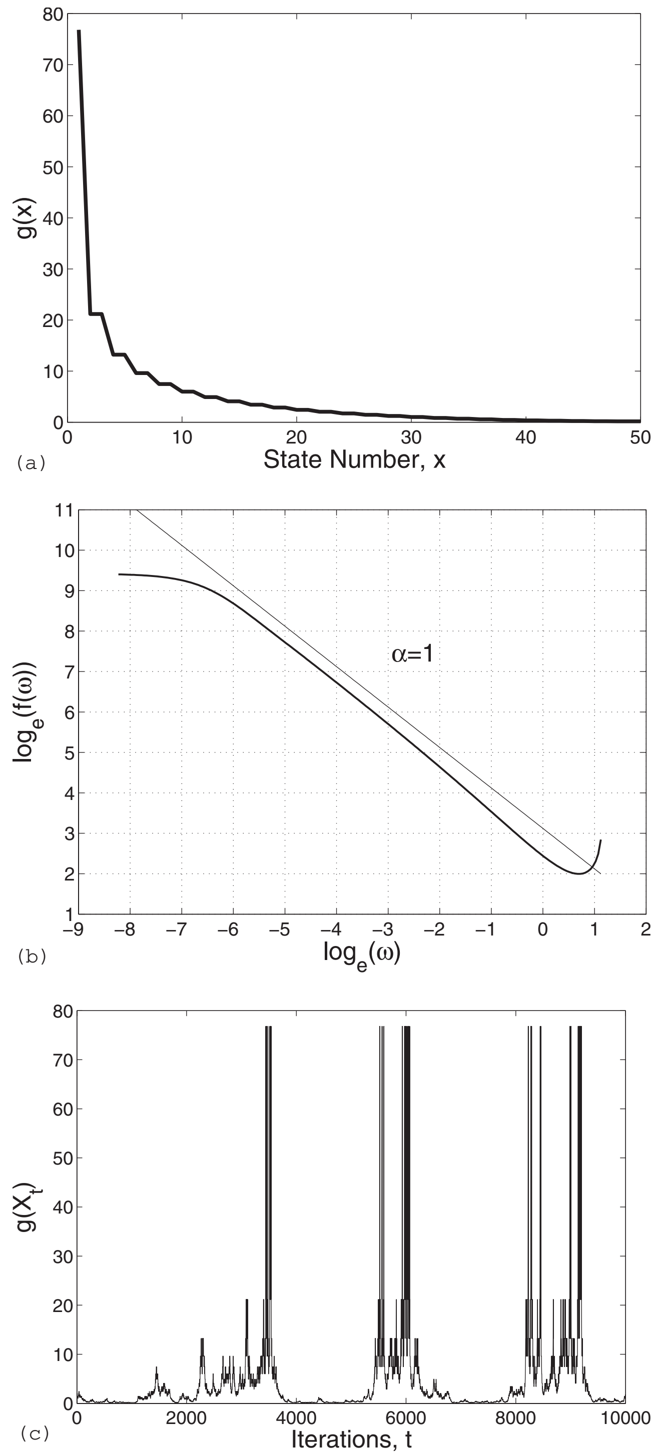

FIG. 1. (a) Values assigned to the states of a random walk according to Proposition 1 with positive $b_{k,m}$'s. The state space is of size $m=50$. (b) Log-log plot of the spectral density of the random walk in (a). The straight line has slope -1. (c) Sample path of the chain.

similarly to an AR1 time series as both are stationary processes with strong similarity to Brownian motion. Let the map $\boldsymbol{g}$ be defined by $g(x)=1/\sqrt{x}$, where g is a function of the underlying process. According to Lemma 3,

$$\begin{aligned} a_k &= |\boldsymbol{g}^T B \boldsymbol{e}_k|^2 \\ &= \left| \frac{\sqrt{2}}{m} \sum_{y=1}^{m} \frac{1}{\sqrt{y}} \cos\left(\frac{k\pi}{m}(y-0.5) \right) \right|^2 \\ &\approx \left| \sqrt{2} \int_0^{\infty} \frac{1}{\sqrt{y}} \cos(\pi x y) dy \right|^2 \\ &= \left| \frac{\sqrt{2}\cos(x/4)}{\sqrt{x}} \right|^2 \equiv a(x), \end{aligned} \tag{22}$$

where $x \equiv k/m$ for $k=1,\ldots,m-1$. The last equality is the Fourier cosine transform of $1/\sqrt{y}$. With the choice of $a(x)$ in (22), (12) in Theorem 1 holds with $\alpha=1$:

$$\frac{a(x)[1-\lambda^{\mathrm{RW}}(x)]}{|\lambda^{\mathrm{RW}\prime}(x)|} = \frac{2\cos^2(x/4)}{|x|} \frac{[1-\cos(\pi x)]}{\pi|\sin(\pi x)|} \xrightarrow{x\to 0} 1.$$

The resulting spectral density is presented in Fig. 2(a).

A $1/\omega$ model which is very similar to these heuristics was also constructed in Sec. IV A. The constructed process took values given by g in (21), and it can be shown that this function is approximately proportional to $1/\sqrt{x}$ for large m.

This indicates that the heuristic explanation can be applied and establishes a connection between the point process and the random walk time series, both producing $1/\omega$ noise.

C. Renewal process

In this section we consider the renewal process of returns to a state i^*. Let $\{X_t^m\}$ be the random walk defined by the transition probability matrix P_{RW}, and let

$$g(i) = \begin{cases} 1, & i = i^* \\ 0, & i = 1,\ldots,i^*-1, \quad i^*+1,\ldots,m, \end{cases} \tag{23}$$

i.e., $g(X_t^{(m)})=1$ every time the chain occupies state i^*, and is zero in all other states. It is well-known that the spectrum of the return times to a central value of a symmetric random walk process with finite variance steps in one dimension, without boundaries, has a regularly varying tail with $f(\omega) \sim 1/\omega^{0.5}$ as $\omega \to 0$([35], p. 431).

Let m be odd. From the eigenvectors of P_{RW} in (17), it follows that for $i^*=(m+1)/2$

$$a_{k,m}^{\mathrm{RW}^*} = \frac{2}{m^2}\cos^2\left(\frac{\pi k}{2}\right) = \begin{cases} \dfrac{2}{m^2}, & k = 2,4,\ldots,m-2 \\ 0, & k = 1,3,\ldots,m-1 \end{cases}$$

for large m. Considering only the even terms of $a_{k,m}^{\mathrm{RW}^*}$, we may choose $a(x)\equiv 2$ and $A_m = 1/m^2$ in Theorem 1. Then (12) holds with $\alpha=0.5$:

$$\frac{a(x)[1-\lambda^{\mathrm{RW}}(x)]^{0.5}}{|\lambda^{\mathrm{RW}\prime}(x)|} = \frac{2[1-\cos(\pi x)]^{0.5}}{\pi|\sin(\pi x)|} \xrightarrow{x\to 0} \frac{\sqrt{2}}{\pi}. \tag{24}$$

Proposition 2. Consider a sequence of renewal processes with an odd number of states m, which take the value 1 when a random walk with transition matrix P_{RW} in (16) occupies state number $(m+1)/2$ and are zero elsewhere. The sequence

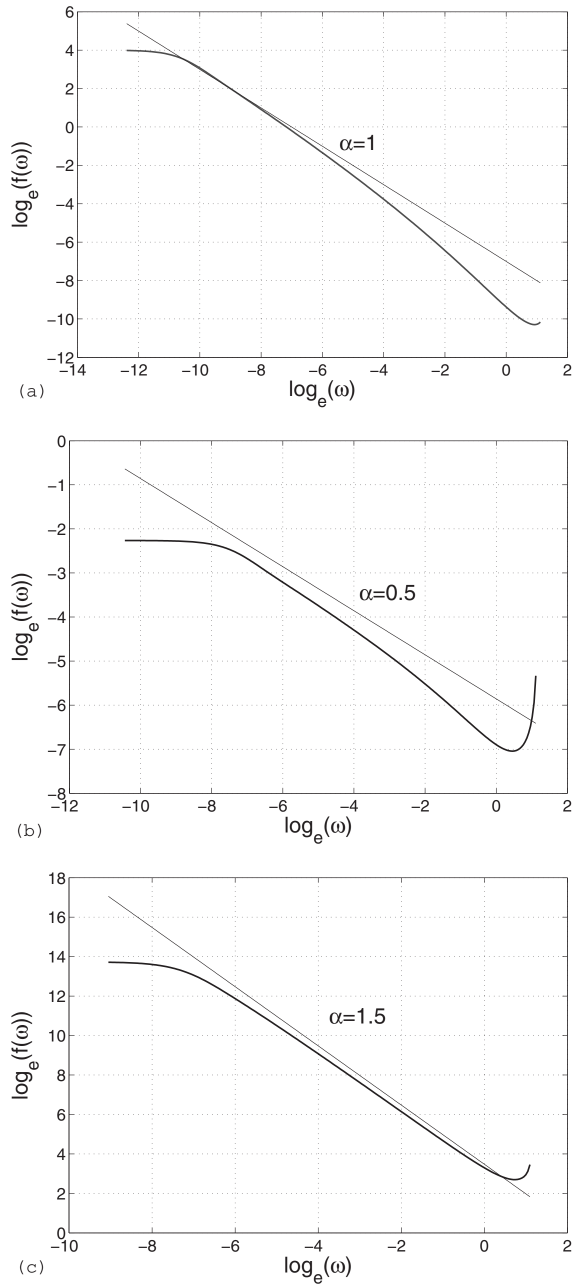

FIG. 2. (a) Log-log plot of the spectral density of the random walk in Sec. IV B with $g(x)=1/\sqrt{x}$. The state space is of size $m=400$. (b) Log-log plot of the spectral density of the return times of a random walk in Proposition 2. The state space is of size $m=151$. (c) Log-log plot of the spectral density of the cyclical random walk in Proposition 3, on a state space of size $m=151$.

produces $1/\omega^{0.5}$ noise for $(1-\lambda_{1,m}^{\mathrm{RW}}) \ll \omega \ll 1$, where the eigenvalues $\{\lambda_{1,m}^{\mathrm{RW}}\}$ are given in Lemma 3. The spectral density is presented in Fig. 2(b) for $m=151$.

D. Cyclical random walk

The boundary conditions of a random walk may have significant influence on the spectral density, as pointed out in [33]. When $\boldsymbol{g}=[1,\ldots,m]^T$, the random walk with reflecting barriers given by P_{RW} exhibits $1/\omega^2$ behavior, as pointed out before Proposition 1. Here, we show that a cyclical random walk with transition matrix P_{Cyc} exhibits $1/\omega^{1.5}$ behavior.

The eigenvalues of P_{Cyc} are $\lambda_{k,m}^{\mathrm{Cyc}}=\lambda^{\mathrm{Cyc}}(k/m)$, where

$$\lambda^{\mathrm{Cyc}}(x) = \cos(2\pi x).$$

If $\boldsymbol{g}=[1,2,\ldots,m]^T$, the coefficients $\{a_{k,m}^{\mathrm{Cyc}}\}$ are given in (18). We see that (12) holds with $a^{\mathrm{Cyc}}(x)=1/4\sin^2(\pi x)$, $A_m\equiv 1$, and $\alpha=1.5$:

$$\frac{a^{\mathrm{Cyc}}(x)[1-\lambda^{\mathrm{Cyc}}(x)]^{1.5}}{|\lambda^{\mathrm{Cyc}\prime}(x)|} = \frac{1}{4\sin^2(\pi x)}\frac{[1-\cos(2\pi x)]^{1.5}}{2\pi|\sin(2\pi x)|} \xrightarrow{x\to 0} \frac{1}{4\sqrt{2}\pi}. \tag{25}$$

Proposition 3. Consider a sequence of cyclical random walks with transition probability matrices P_{Cyc} in (16), and eigenvalues $\{\lambda_{k,m}^{Cyc}\}$ given in Lemma 3. When $\boldsymbol{g}=[1,\ldots,m]^T$, the sequence produces $1/\omega^{1.5}$ noise for $(1-\lambda_{1,m}^{Cyc})\ll\omega\ll 1$. The spectral density is presented in Fig. 2(c) for $m=151$.

E. Metropolized independence sampler

The Metropolized independence sampler (MIS) is a simple version of the Metropolis-Hastings algorithm. Following [26], we let the m vector $\boldsymbol{q}$ be a proposal distribution and order the states such that $\frac{\pi(1)}{q(1)}>\frac{\pi(2)}{q(2)}>\cdots>\frac{\pi(m)}{q(m)}$, where $\boldsymbol{\pi}$ is the desired stationary distribution. The $m\times m$ transition probability matrix of the MIS is given by

$$P_{\mathrm{MIS}}(x,y) = \begin{cases} q(y), & y<x \\ q(X\geq x) - \dfrac{q(x)}{\pi(x)}\pi(X>x), & y=x \\ \pi(y)\dfrac{q(x)}{\pi(x)}, & y>x. \end{cases}$$

If the stationary distribution $\boldsymbol{\pi}$ is uniform, then for any proposal distribution, $\boldsymbol{q}$, states with higher values are less stable. That is, the probability of remaining in a state x decreases with increasing x; but given that a move occurs, the probability of taking a big leap is higher the more stable x is.

Assume that the stationary distribution $\boldsymbol{\pi}$ is uniform, the proposal distribution $\boldsymbol{q}$ is strictly increasing, and that $\boldsymbol{g}^T=[1,\ldots,m]/m$. Applying the eigendecomposition of a MIS in [26], we have

$$\lambda_{k,m}^{\mathrm{MIS}} = 1 - q(X\leq k) - q(k)(m-k), \tag{26}$$

$$a_{k,m}^{\mathrm{MIS}} = \frac{(m-k)(m-k+1)}{4m^3}, \tag{27}$$

for $k=1,\ldots,m-1$. If $\{\lambda_{k,m}^{\mathrm{MIS}}\}_{k=1}^{m-1}$ is a given sequence of eigenvalues, we see from [26] that a corresponding proposal distribution, $\boldsymbol{q}$, will satisfy the iterative sequence of equations:

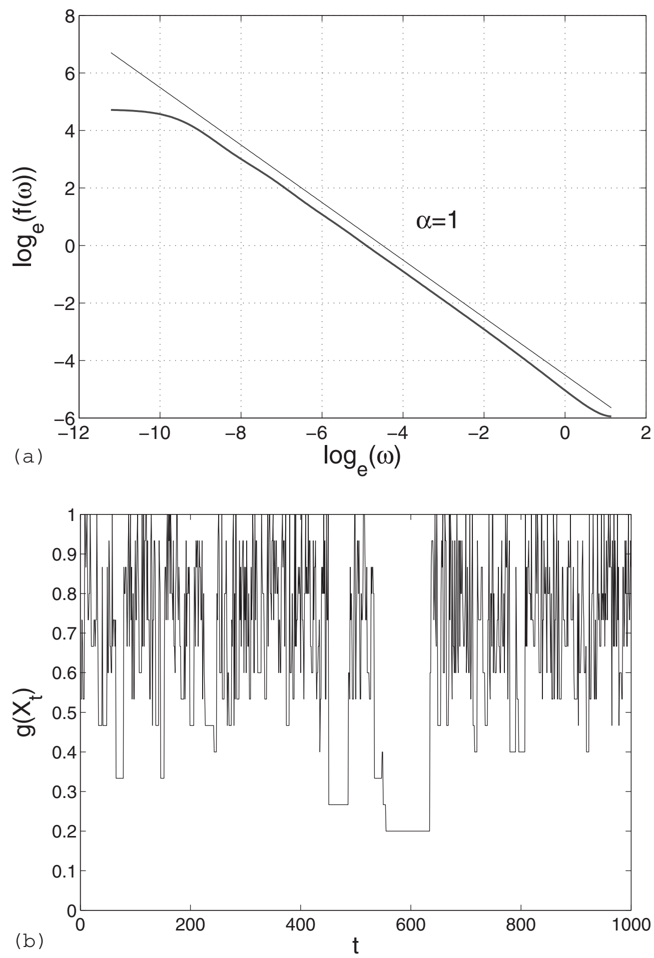

FIG. 3. (a) Log-log plot of the spectral density of a Metropolized independence sampler (MIS2) on a state space of size $m=15$. The eigenvalues are given by (30), with $\log(1-\lambda_{1,15})=\log(1-0.9999)=-9.21$, and $\boldsymbol{g}=[1,\ldots,m]/m$. The straight line has slope -1. (b) Sample path of the chain.

$$q(k) = \frac{1}{m-k+1}\left(1-\lambda_{k,m}^{\mathrm{MIS}} - \sum_{j=1}^{k-1} q(j)\right), \tag{28}$$

for $k=1,\ldots,m$, where $\lambda_{m,m}^{\mathrm{MIS}}\equiv 0$. This enables us to design a MIS with any desired sequence of decreasing positive eigenvalues.

From [27], $a_{k,m}^{\mathrm{MIS}}\approx(m-k)^2/4m^3$ for large m, such that we may choose $a^{\mathrm{MIS}}(x)=(1-x)^2/4$ and $A_m=1/m$ in Theorem 1. In order to satisfy (12) and obtain $1/\omega^{\alpha}$ noise, it is sufficient to let $\lambda(x)$ be the solution of $a^{\mathrm{MIS}}(x)=C|\lambda'(x)|/[1-\lambda(x)^{\alpha}]$. Then

$$\lambda^{\mathrm{MIS1}}(x) = \begin{cases} 1-\{1-[1-(1-\lambda^*)^{1-\alpha}](1-x)^3\}^{1/(1-\alpha)} \\ 1-(1-\lambda^*)^{(1-x)^3} \end{cases} \tag{29}$$

for $\alpha\neq 1$ and $\alpha=1$, respectively. A MIS with a sequence of eigenvalues taking the values of this function can be constructed by choosing $\boldsymbol{q}$ as in (28), with $\lambda_{k,m}^{\mathrm{MIS}}=\lambda^{\mathrm{MIS1}}[1-(m-k)/(m-1)]$, such that $\lambda_{1,m}^{\mathrm{MIS}}=\lambda^*=\lambda^{\mathrm{MIS1}}(0)$.

Another way to satisfy (12) of Theorem 1 is to let $\lambda(x)$ be the solution of $a^{\mathrm{MIS}}(x)=C|\lambda'(x)|/\{[1-\lambda(x)]^{\alpha}[1+\lambda(x)]\}$. For $\alpha=1$ we have

$$\lambda^{\mathrm{MIS2}}(x)=\tanh\{[\tanh^{-1}(\lambda^{*})](1-x)^{3}\}. \qquad (30)$$

$\boldsymbol{q}$ can be constructed in the same manner as for $\lambda^{\mathrm{MIS1}}(x)$.

We construct the eigenvalue sequences such that $\lambda^{\mathrm{MIS1}}(1)=0=\lambda^{\mathrm{MIS2}}(1)$, but other, positive, values are possible. Figure 3 illustrates the result for MIS2 with $m=15$. The spectrum clearly exhibits $1/\omega$ behavior for a wide range of $\log(\omega)$ with change in behavior near $\omega=1-\lambda_{1,15}$. However, strictly speaking, Theorem 1 cannot be applied since (29) and (30) are well-defined only for $\lambda^{\mathrm{MIS1}}(0)=\lambda^{*}=\lambda^{\mathrm{MIS2}}(0)<1$.

A limit chain can be constructed for a sequence of MIS chains obtained by choosing the eigenvalues according to (29) or (30). Let the range of $\boldsymbol{g}^{T}$ be $[1,2,\ldots,m]/m$. The limiting state space can be taken as the interval (0, 1]. The probability densities $q(x)$ and $\pi(x)$ replace $\boldsymbol{q}$ and $\boldsymbol{\pi}$, respectively in a corresponding algorithm: A new proposal Y_{t+1} from q is accepted as X_{t+1} with probability $\min[1,\pi(Y_{t+1})q(X_t)/\pi(X_t)q(Y_{t+1})]$, while $X_{t+1}=X_t$ elsewhere.

Set $k=\lceil x(m-1)\rceil$ for $x\in(0,1)$ and let $m\to\infty$ in (26). By differentiating (26), the limiting relation between $\boldsymbol{q}$ and $\{\lambda_{k,m}^{\mathrm{MIS}}\}$ becomes

$$\frac{dq}{dx}=\frac{-1}{(1-x)}\frac{d\lambda^{\mathrm{MIS}}}{dx}, \qquad (31)$$

with initial condition $q(0)=1-\lambda^{\mathrm{MIS}}(0)$. The desired limiting eigenvalue functions in (29) and (30) are obtained if $q(x)$ is either of the probability densities

$$q^{\mathrm{MIS1}}(x)=\begin{cases}(1-\lambda^{*})+\displaystyle\int_0^x\frac{3\kappa_1}{1-\alpha}(1-y)[1-\kappa_1(1-y)^3]^{\alpha/(1-\alpha)}dy, & \alpha\neq 1\\ (1-\lambda^{*})-\displaystyle\int_0^x 3\kappa_2(1-y)e^{\kappa_2(1-y)^3}dy, & \alpha=1\end{cases} \qquad (32)$$

or

$$q^{\mathrm{MIS2}}(x)=(1-\lambda^{*})+\int_0^x 3\kappa_3(1-y)\mathrm{sech}^2[\kappa_3(1-y)^3]dy, \qquad (33)$$

respectively, where $\kappa_1=1-(1-\lambda^{*})^{1-\alpha}$, $\kappa_2=\log(1-\lambda^{*})$, and $\kappa_3=\tanh^{-1}(\lambda^{*})$.

Proposition 4. Consider a sequence of MIS chains with uniform stationary density $\pi(x)$ on the continuous state space (0, 1]. Let the proposal densities be given by either (32) or (33), with $\alpha\in(0,2)$ and $\alpha=1$, respectively, and $\lambda^{*}=\lambda_n^{*}$. Then the MIS sequence produces $1/\omega^{\alpha}$ noise for $1-\lambda_n^{*}\ll\omega\ll 1$.

Notice that if $\lambda_n^{*}\equiv 1$, $q^{\mathrm{MIS1}}(x)$ and $q^{\mathrm{MIS2}}(x)$ become degenerate when $\alpha\geq 1$, but if $\alpha<1$, the densities exist and the limiting spectral densities are asymptotically proportional to $1/\omega^{\alpha}$ as $\omega\to 0$.

V. DISCUSSION

As we mentioned in the Introduction, $1/\omega^{\alpha}$ noise is observed in many contexts. There have been a number of attempts to explain the generality of this phenomenon, and we have cited several of these in our bibliography. None has been sufficiently general to cover the broad range of contexts where $1/\omega^{\alpha}$ noise is seen. This paper considerably broadens the scope of possible applications, as we have indicated by various examples. However, we have not attempted here to describe how our results apply in data contexts where $1/\omega^{\alpha}$ noise is observed. This will be a large additional challenge. For example $1/\omega^{\alpha}$ noise is observed in membrane channel currents, and synthetic channels, nonopores, have been created to enable measurements which are impossible to obtain in biological materials [36]. It is found [36] that alternation between open and closed states in voltage-gated potassium channels can produce a process with a striking $1/\omega^{\alpha}$ power spectral density. Although this data may have a non-Markov character, as suggested in [36], our results indicate that a Markov model might well explain this property of the data. However, we will not attempt to formulate the details here.

This paper has been about discrete time processes. However, consider a continuous time process on discrete state space, defined by a *probability rate matrix* R. Let

$$R=\frac{2}{\tau}(P-I),$$

where P is a probability transition matrix with zero diagonal, τ is a time constant, and the eigenvalues of R, $\gamma_{k,m}=2(\lambda_{k,m}-1)/\tau$ are negative if $\{\lambda_{k,m}\}$ are real. R and P have the same eigenvectors. For continuous time processes, Reynolds [28] presents the covariance function,

$$C(t)=\sum_{k=1}^{m-1}a_k e^{t\gamma_{k,m}},$$

and the spectral density for continuous time processes is given by

$$f_m(\omega) = \frac{1}{\pi}\sum_{k=1}^{m-1} a_{k,m}\frac{-\gamma_{k,m}}{\gamma_{k,m}^2+\omega^2}, \quad \omega \in (0,\infty). \tag{34}$$

As with the discrete case, we can approximate this with an integral, but the cosine-term in the discrete time expression disappears, and the conditions for $1/\omega^{\alpha}$-behavior are simpler to obtain. This is shown heuristically for aggregation of relaxation processes in, e.g., [7]. Formally, for finite state Markov chains, we may let $a(x)$ be as in Theorem 1, and assume that $\gamma_{k,m}=\gamma(k/m)$, where $\gamma(x)$ is a continuously differentiable real function. If there exists a constant C such that

$$a(x) = \frac{C|\gamma'(x)|}{[-\gamma(x)]^{\alpha}},$$

for $\alpha \in (0,2)$, then (34) can be treated as a Rieman sum, such that

$$f_m(\omega) \approx \frac{1}{\omega^{\alpha}}\frac{1}{\pi}\int_{-\gamma_{1,m}/\omega}^{-\gamma_{m-1,m}/\omega}\frac{y^{1-\alpha}}{y^2+1}dy$$
$$\sim \begin{cases} 1 & \text{for } 0 \ll \omega \ll -\gamma_{1,m}, \\ \dfrac{1}{\omega^{\alpha}} & \text{for } -\gamma_{1,m} \ll \omega \ll -\gamma_{m-1,m} \\ \dfrac{1}{\omega^{2}} & \text{for } -\gamma_{m-1,m} \ll \omega \ll \infty. \end{cases},$$

Since we can tune the parameter τ, the interval of $1/\omega^{\alpha}$-behavior can be stretched in both directions to 0 and ∞.

The work of P. E. Greenwood was supported by NSERC Canada. The collaboration was made possible due to the generous hospitality of Arizona State University and Stockholm University.

APPENDIX A: PROOFS

Proof of Lemmas 1 and 2. At equilibrium

$$\begin{aligned}\mathrm{Cov}_{\pi}[g(X_0),g(X_t)] &= \sum_x\sum_y g(x)g(y)\pi(x)P^t(x,y) \\ &\quad - \sum_x g(x)\pi(x)\sum_y g(y)\pi(y) \\ &= \boldsymbol{g}^T B P^t \boldsymbol{g} - \boldsymbol{g}^T\boldsymbol{\pi}\boldsymbol{\pi}^T\boldsymbol{g} = \boldsymbol{g}^T B(P^t-\Pi)\boldsymbol{g} \\ &= \boldsymbol{g}^T B\left(\sum_{k=1}^{m-1}\lambda_k^t \boldsymbol{e}_k \boldsymbol{f}_k^T\right)\boldsymbol{g} = \sum_{k=1}^{m-1} a_k\lambda_k^t, \end{aligned} \tag{A1}$$

where $a_k=(\boldsymbol{g}^T B\boldsymbol{e}_k)(\boldsymbol{f}_k^T\boldsymbol{g})$, and $\boldsymbol{g}$ is an m vector that assigns real values to the states in the abstract state space. We use that $B\Pi=B\mathbf{1}\boldsymbol{\pi}^T=\boldsymbol{\pi}\boldsymbol{\pi}^T$ in the third line, and the eigendecomposition (5) in the fourth line. Here, $\mathbf{1}\equiv[1,\ldots,1]^T$. The spectral density (7) is now easily found by taking the Fourier transform of each of the terms in Eq. (A1). Lemma 2 is proved along the same lines.

Proof of Theorem 1. For large m, the spectral density $f_m(\omega)$ in (7) is near to an integral:

$$\begin{aligned} f_m(\omega) &= \frac{1}{\pi}\sum_{k=1}^{m-1}\frac{a_{k,m}(1-\lambda_{k,m}^2)}{1+\lambda_{k,m}^2-2\lambda_{k,m}\cos(\omega)} \\ &= \frac{1}{\pi}\sum_{k=1}^{m-1}\frac{m a_{k,m}(1-\lambda_{k,m}^2)}{(1-\lambda_{k,m})^2+2\lambda_{k,m}[1-\cos(\omega)]}\frac{1}{m} \\ &= \frac{1}{\pi}\sum_{k=1}^{m-1}\frac{mA_m a\left(\frac{k}{m}\right)\left[1-\lambda^2\left(\frac{k}{m}\right)\right]}{\left[1-\lambda\left(\frac{k}{m}\right)\right]^2+2\lambda\left(\frac{k}{m}\right)[1-\cos(\omega)]}\frac{1}{m} \\ &\approx \frac{1}{\pi}\int_{1/m}^{(m-1)/2}\frac{mA_m a(x)[1-\lambda^2(x)]}{[1-\lambda(x)]^2+2\lambda(x)[1-\cos(\omega)]}dx \\ &\equiv \tilde{f}_m(\omega). \end{aligned} \tag{A2}$$

We first show that $\tilde{f}_m(\omega)$ is asymptotically proportional in log scale to $1/\omega^{\alpha}$ on the interval (L_m,H_m) (see Definition 1). Suppose $\{\omega_m\}$ is a sequence such that $L_m/\omega_m\to 0$ and $H_m/\omega_m\to\infty$. Set $D_m(x)=mA_m a(x)[1-\lambda(x)]^{\alpha}[1+\lambda(x)]/|\lambda'(x)|$, and substitute $\lambda=\lambda(x)$ [with $x(\lambda)$ denoting the inverse of $\lambda(x)$] in Eq. (A2),

$$\tilde{f}_m(\omega_m) = \frac{1}{\pi}\int_{\lambda_{m-1,m}}^{\lambda_{1,m}}\frac{(1-\lambda)^{1-\alpha}}{(1-\lambda)^2+2\lambda[1-\cos(\omega_m)]}D_m[x(\lambda)]d\lambda \tag{A3}$$

$$\begin{aligned} &= \frac{1}{\pi}\int_{L_m}^{H_m}\frac{z^{1-\alpha}}{z^2+2(1-z)[1-\cos(\omega_m)]} \\ &\quad \times D_m[x(1-z)]dz \\ &= \frac{1}{\omega_m^{\alpha}\pi}\int_{L_m/\omega_m}^{H_m/\omega_m}\frac{y^{1-\alpha}}{y^2+\frac{2}{\omega_m^2}[1-\cos(\omega_m)](1-y\omega_m)} \\ &\quad \times D_m[x(1-y\omega_m)]dy, \end{aligned} \tag{A4}$$

where $\lambda=1-z$ and $z=y\omega_m$. Since $\omega_m\to 0$ when $H_m/\omega_m\to\infty$, and $\lambda(x)$ is a decreasing function, $x(1-y\omega_m)\to 0$. By assumption (12) we also have that

$$\frac{D_m(x)}{mA_m} = \frac{a(x)[1-\lambda(x)]^{\alpha}}{|\lambda'(x)|}[1+\lambda(x)] \overset{x\to 0}{\to} 2C.$$

Since $D_m(x)/(mA_m)$ is bounded by assumption, the integrand of Eq (A4) is bounded by a multiple of $y^{1-\alpha}/(y^2+1)$ for $\omega<1/4$, and the dominated convergence theorem gives that

$$\frac{\omega_m^{\alpha}\tilde{f}_m(\omega_m)}{mA_m} \overset{m\to\infty}{\to} \frac{2C}{\pi}\int_0^{\infty}\frac{y^{1-\alpha}}{y^2+1}dy.$$

This implies that $\tilde{f}_m(\omega)\sim 1/\omega_m^{\alpha}$ for $L_m\ll\omega\ll H_m$, according to Definition 1, where we choose $c_m=\pi/\left(2CmA_m\int_0^{\infty}\frac{y^{1-\alpha}}{y^2+1}dy\right)$.

Now we show that $f_m(\omega)$ also is proportionally asymptotic in log scale to $1/\omega^{\alpha}$ for $L_m\ll\omega\ll H_m$. Specifically, we show that the approximation in Eq. (A2) converges in the sense,

$$c_m\omega_m^{\alpha}|f_m(\omega_m)-\tilde{f}_m(\omega_m)| \overset{m\to\infty}{\to} 0,$$

for all sequences $\{\omega_m\}$ such that $L_m/\omega_m\to 0$ and $H_m/\omega_m\to\infty$. Of course, for fixed ω, the sum $f_m(\omega)$ will converge to the integral $\tilde{f}_m(\omega)$. We will show that the distance decreases when $m\to\infty$, even when ω varies. The integrand of $\tilde{f}_m(\omega)$ in Eq. (A2) can be written as

$$I_m(x;\omega)=\frac{[1-\lambda(x)]^{1-\alpha}|\lambda'(x)|}{[1-\lambda(x)]^2+2\lambda(x)[1-\cos(\omega)]}\times\frac{mA_m a(x)[1-\lambda(x)]^{\alpha}[1+\lambda(x)]}{|\lambda'(x)|}. \quad (A5)$$

We need to divide the interval of integration into three parts. In the two first parts we may use the common integral convergence test for series, which states that the distance between the integral and the sum is bounded by the maximum term in the series when the integrand is monotone. In the third part the distance does not depend on ω.

Note that $\omega_m\to 0$, so it is sufficient to consider the case when $\omega\approx 0$. First, consider the part of the interval of integration where $x\approx 0$, and use that $\lambda(x)=1+\lambda^{(r)}(0)x^r/r! + \mathcal{O}(x^{r+1})$ for some $r\geq 1$. Then $\lambda'(x)=r[\lambda(x)-1]/x+\mathcal{O}(x^r)$, and according to condition (12),

$$\begin{aligned}I_m(x;\omega)&\approx\frac{[1-\lambda(x)]^{1-\alpha}|\lambda'(x)|}{[1-\lambda(x)]^2+\omega^2}2CmA_m\\&\approx\frac{[1-\lambda(x)]^{2-\alpha}}{[1-\lambda(x)]^2+\omega^2}\frac{2CmA_m r}{x}\\&\approx 2CmA_m r\frac{[-\lambda^{(r)}(0)/r!]^{2-\alpha}x^{(2-\alpha)r-1}}{[-\lambda^{(r)}(0)x^r/r!]^2+\omega^2}. \quad (A6)\end{aligned}$$

For $(2-\alpha)r\leq 1$, the right-hand side of the last line in Eq. (A6) is decreasing. For $(2-\alpha)r>1$, the right-hand side of the last line in Eq. (A6) is increasing for $x\in(0,x_\omega^*)$, where the maximum point x_ω^* is given by

$$\frac{-\lambda^{(r)}(0)}{r!}x_\omega^{*r}=\sqrt{\frac{(2-\alpha)r-1}{\alpha r+1}}\omega=K\omega. \quad (A7)$$

(This representation will be useful later.) When ω is small, x_ω^* is also small. Moreover, when $x_\omega^*<x\ll 1$, the last part of Eq. (A5) is approximately constant, and $I(x;\omega)$ behaves like $1/\{x[1-\lambda(x)]^{\alpha}\}$, and is decreasing for $x\in(x_\omega^*,x_c)$ for any r and α. The value of x_c depends on the last factor of Eq. (A5). For $x\in(x_c,1)$, the dependence of the distance on ω vanishes when $\omega\to 0$. Hence we divide the interval of integration into the intervals $(0,x_\omega^*)$, (x_ω^*,x_c), and $(x_c,1)$, where we set $x_\omega^*\equiv 0$ if $(2-\alpha)r\leq 1$.

The integral test for series gives that for $x\in(0,x_c)$, the distance between the sum and the integral is bounded by $\max\{I_m(1/m;\omega)/m, 2I_m(x_\omega^*;\omega)/m\}$. Consequently,

$$c_m\omega_m^{\alpha}|f_m(\omega_m)-\tilde{f}_m(\omega_m)|\leq c_m\omega_m^{\alpha}[I_m(1/m;\omega_m)/m + 2I_m(x_{\omega_m}^*;\omega_m)/m+R_m mA_m],$$

where $\{R_m\}$ is a sequence decreasing independently of ω_m, and $c_m=\pi/\left(2CmA_m\int_0^\infty\frac{y^{1-\alpha}}{y^2+1}dy\right)$. The two first terms on the right-hand side also goes to 0: From Eq. (A6),

$$\lim_{m\to\infty}c_m\omega_m^{\alpha}I_m\left(\frac{1}{m};\omega_m\right)\frac{1}{m}=\lim_{m\to\infty}2c_mCmA_m r\frac{(1-\lambda_{1,m})^{2-\alpha}}{\omega_m^{2-\alpha}}=0,$$

$$\lim_{m\to\infty}c_m\omega_m^{\alpha}I_m(x_{\omega_m}^*;\omega_m)\frac{1}{m}=\lim_{m\to\infty}2c_mCmA_m r\frac{\omega_m^{2-\alpha}}{2\omega_m^{2-\alpha}}\frac{\frac{1}{m}}{x_{\omega_m}^*}=0. \quad (A8)$$

We use that $\lambda(1/m)=\lambda_{1,m}$, and $(1-\lambda_{1,m})/\omega_m\to 0$. In Eq. (A8), $(1/m)/x_{\omega_m}^*\to 0$, since $[1-\lambda(1/m)]/[1-\lambda(x_{\omega_m}^*)]\approx(1-\lambda_{1,m})/(K\omega_m)\to 0$ from Eq. (A7), and $\lambda(x)$ is decreasing in x. Hence we have showed that the integral approximates the sum in an appropriate way, and $f_m(\omega)\sim 1/\omega^{\alpha}$ for $L_m\ll\omega\ll H_m$.

The asymptotic behavior of $f_m(\omega)$ for $0\ll\omega\ll L_m$ and $H_m\ll\omega\ll 1$ is easy to show, since all the terms in the spectral density in (7) have the desired asymptotic behavior on these intervals.

Proof of Corollary 1. We apply results from the proof of Theorem 1. The spectral density of an AR1-process is given by (10). The aggregation of such processes in (14) with rates having probability density $h(\beta)$ has spectral density

$$\begin{aligned}f(\omega_n)&=\frac{\sigma^2}{\pi}\int_0^1\frac{h(\beta)}{1+\beta^2-2\beta\cos(\omega_n)}d\beta\\&=\frac{\sigma^2}{\pi}\int_0^1\frac{(1-\beta)^{1-\alpha}}{(1-\beta)^2-2\beta[1-\cos(\omega_n)]}\frac{h(\beta)}{(1-\beta)^{1-\alpha}}d\beta.\end{aligned}$$

Since $h(\beta)/(1-\beta)^{1-\alpha}\to C$ as $\beta\to 1$, Eqs. (A3) and (A4) in the proof of Theorem 1 give that

$$\omega_n^{\alpha}f(\omega_n)\to\frac{C\sigma^2}{\pi}\int_0^\infty\frac{y^{1-\alpha}}{y^2+1}dy$$

for any sequence $\omega_n\to 0$. Consequently, $f(\omega)\sim 1/\omega^{\alpha}$ for $0\ll\omega\ll 1$.

Proof of Corollary 2. We apply results from the proof of Theorem 1. The spectral density is given by

$$f_n(\omega_n)=\frac{\sigma^2}{\pi c_n}\int_{\beta_n^{\min}}^{\beta_n^{\max}}\frac{(1-\beta)^{1-\alpha}}{(1-\beta)^2-2\beta[1-\cos(\omega_n)]}\frac{h(\beta)}{(1-\beta)^{1-\alpha}}d\beta,$$

where $c_n=\int_{\beta_n^{\min}}^{\beta_n^{\max}}h(\beta)d\beta$. From Eqs. (A3) and (A4) in the proof of Theorem 1,

$$\omega_n^{\alpha}c_nf_n(\omega_n)\to\frac{C\sigma^2}{\pi}\int_0^\infty\frac{y^{1-\alpha}}{y^2+1}dy,$$

for all sequences $\{\omega_n\}$ which satisfy $(1-\beta_n^{\max})/\omega_n\to 0$ and $\omega_n/(1-\beta_n^{\min})\to 0$.

Proof of Lemma 3. Feller [22] presents a proof for the more general cases, where the probability of moving to each of the neighboring states can be different. He also states the result for general cyclical matrices which includes our P_{Cyc}. It is straightforward to verify that the eigenvectors and stationary distributions are correct.

Proof of Proposition 1. We apply Theorem 1. $\lambda(x) \equiv \lambda^{\mathrm{RW}}(x)=\cos(\pi x)$. Since $(\boldsymbol{e}_k^T B \boldsymbol{e}_j)^2$ is 1 for $j=k$ and 0 elsewhere for reversible chains,

$$a_{k,m} = (\boldsymbol{e}_{k,m}^T B \boldsymbol{g})^2 = \left((\boldsymbol{e}_{k,m}^{\mathrm{RW}})^T B \sum_{j=1}^{m-1} b_{j,m} \boldsymbol{e}_{j,m}^{\mathrm{RW}} \right)^2 = b_{k,m}^2,$$

and $a_{k,m}=a(k/m)$ in (20) by the choice of $b_{k,m}$. In Theorem 1 $A_m \equiv 1$ and (12) is satisfied with $C=1$ since $a(x)$ in (20) satisfies (13). That $a(x)[1+\lambda^{\mathrm{RW}}(x)][1-\lambda^{\mathrm{RW}}(x)]^{\alpha}/|\lambda^{\mathrm{RW}\prime}(x)| = 1+\lambda^{\mathrm{RW}}(x)$ is bounded, follows straightforwardly.

Proof of Proposition 2. We apply Theorem 1. $\lambda(x) \equiv \lambda^{\mathrm{RW}}(x)=\cos(\pi x)$. For even k, $a_{k,m}=A_m a(k/m)$, with $A_m = 1/m^2$ and $a(x)\equiv 2$. For odd k, $a_{k,m}=0$, but this will only affect the Rieman sum approximation in the proof of Theorem 1 by a factor of 2. Finally, condition (12) is satisfied in (24) with $C=\sqrt{2}/\pi$. That $a(x)[1+\lambda^{\mathrm{RW}}(x)][1-\lambda^{\mathrm{RW}}(x)]^{0.5}/|\lambda^{\mathrm{RW}\prime}(x)| = 2\sqrt{1+\lambda^{\mathrm{RW}}(x)}/\pi$ is bounded, follows straightforwardly.

Proof of Proposition 3. We apply Theorem 1. $a_{k,m} = A_m a^{\mathrm{Cyc}}(k/m)$, with $A_m \equiv 1$ and $a^{\mathrm{Cyc}}(x)=1/[4\sin^2(\pi x)]$. $\lambda(x)\equiv\lambda^{\mathrm{Cyc}}(x)=\cos(2\pi x)$. $\lambda^{\mathrm{Cyc}}(x)$ is only decreasing for $x\in(0,0.5)$, but notice that both $\lambda_{k,m}^{\mathrm{Cyc}}=\lambda_{m-k,m}^{\mathrm{Cyc}}$ and $a_{k,m}^{\mathrm{Cyc}} = a_{m-k,m}^{\mathrm{Cyc}}$. Hence in the Rieman sum approximation in the proof of Theorem 1, it is sufficient to multiply with a factor of 2 and integrate over $x\in(0,0.5)$. On this interval $\lambda^{\mathrm{Cyc}}(x)$ is decreasing and invertible. Finally, condition (12) is satisfied in (25) with $C=1/(4\sqrt{2}\pi)$. That $a(x)[1+\lambda^{\mathrm{Cyc}}(x)][1-\lambda^{\mathrm{Cyc}}(x)]^{1.5}/|\lambda^{\mathrm{Cyc}\prime}(x)| = \sqrt{1+\lambda^{\mathrm{Cyc}}(x)}/(4\pi)$ is bounded, follows straightforwardly.

Proof of Proposition 4. We apply results from the proof of Theorem 1. The relation between the eigenvalues and the proposal distribution of a MIS with uniform stationary distribution is given in (26). As $m\to\infty$, this equation turns into

$$\lambda^{\mathrm{MIS}}(x) = 1 - F_q(x) - q(x)(1-x),$$

where $q(x)$ is a probability density and $F_q(x)$ is the corresponding cumulative distribution. Both are differentiable functions if $\lambda^{\mathrm{MIS}}(x)$ is differentiable. Differentiating on both sides, we get (31). The proposal densities in (32) and (33) are the solutions of (31), when $\lambda^{\mathrm{MIS}}(x)$ is one of the differentiable functions (29) and (30), respectively. Moreover, $a_{k,m}\approx A_m a(k/m)$ for large m, where $A_m=1/m$, and $a(x) = (1-x)^2/4$.

Now, $a(x)=|\lambda'(x)|/[1-\lambda(x)]^{\alpha}$, and $a(x)=|\lambda'(x)|/\{[1-\lambda(x)]^{\alpha}[1+\lambda(x)]\}$ for MIS1 and MIS2, respectively. The spectral density of the limiting MIS-sequences with $\lambda^{\mathrm{MIS}}(0)=\lambda_n^*$ and $\lambda^{\mathrm{MIS}}(1)=0$ is given by Eq. (A2), and it becomes

$$f_n(\omega) = \frac{1}{\pi}\int_0^1 \frac{a(x)[1-\lambda^2(x)]}{[1-\lambda(x)]^2 + 2\lambda(x)[1-\cos(\omega)]}dx$$
$$= \frac{1}{\pi}\int_0^{\lambda_n^*} \frac{(1-\lambda)^{1-\alpha}}{(1-\lambda)^2+2\lambda[1-\cos(\omega)]} D[\lambda^{-1}(\lambda)]d\lambda,$$

where $D[\lambda^{-1}(\lambda)]=1+\lambda$ for MIS1 and $D[\lambda^{-1}(\lambda)]=1$ for MIS2. From Eqs. (A3) and (A4) in the proof of Theorem 1, $f_n(\omega)\sim 1/\omega^{\alpha}$ for $(1-\lambda_n^*)\ll\omega\ll 1$.

[1] J. Beran, *Monographs on Statistics and Applied Probability* (Chapman and Hall, New York, 1994), Vol. 61.

[2] J. Davidsen and H. G. Schuster, Phys. Rev. E **65**, 026120 (2002).

[3] M. S. Keshner, Proc. IEEE **70**, 212 (1982).

[4] C. W. J. Granger, J. Econom. **14**, 227 (1980).

[5] F. Ramirez-Perez and R. Serfling, Adv. Appl. Probab. **33**, 631 (2001).

[6] N. J. Kasdin, Proc. IEEE **83**, 802 (1995).

[7] E. Milotti, Technical Report, Università di Udine, Italy, 2002, e-print arXiv:physics/0204033v1 (unpublished).

[8] B. B. Mandelbrot and J. W. Van Ness, SIAM Rev. **10**, 422 (1968).

[9] G. W. Wornell, Proc. IEEE **81**, 1428 (1993).

[10] M. B. Weissman, Rev. Mod. Phys. **60**, 537 (1988).

[11] B. Kaulakys and T. Meskauskas, Phys. Rev. E **58**, 7013 (1998).

[12] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).

[13] F. Gruneis, Fluct. Noise Lett. **1**, R119 (2001).

[14] S. B. Lowen and M. C. Teich, Phys. Rev. E **47**, 992 (1993).

[15] E. Marinari, G. Parisi, D. Ruelle, and P. Windey, Phys. Rev. Lett. **50**, 1223 (1983).

[16] M. Usher, M. Stemmler, and Z. Olami, Phys. Rev. Lett. **74**, 326 (1995).

[17] B. Kaulakys, V. Gontis, and M. Alaburda, Phys. Rev. E **71**, 051105 (2005).

[18] S.-L. Kim, J. Jeoung, Y. Kwan, Y. I. Kim, S. H. Jung, and K. J. Lee, J. Comput. Neurosci. **19**, 39 (2005).

[19] G. Zanella, Technical Report, Istituto Nazionale di Fisica Nucleare, Italy, 2005, e-print arXiv:physics/0504164v2 (unpublished).

[20] C. W. J. Granger and R. Joyeux, J. Time Ser. Anal. **1**, 15 (1980).

[21] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Encyclopedia of Mathematics and its Applications* (Cambridge University Press, Cambridge, 1989), Vol. 27.

[22] W. Feller, *An Introduction to Probability Theory and its Applications*, 3rd ed. (Wiley, New York, 1968), Vol. I.

[23] S. Erland, Ph.D. thesis, Norwegian University of Science and Technology, 2003 (unpublished).

[24] P. Diaconis and P. Hanlon, in *Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications* (American Mathematical Society, Providence, RI, 1992), pp. 99–117.

[25] P. J. Green and X. L. Han, in *Stochastic Models, Statistical Methods and Algorithms in Image Analysis*, edited by P. Barone, A. Frigessi, and M. Piccioni, No. 74 in Lecture Notes in Statistics (Springer, Berlin, 1992), pp. 142–164.

[26] J. S. Liu, Stat. Comput. **6**, 113 (1996).

[27] T. W. Anderson, in *Probability, Statistics, and Mathematics* (Academic Press, Boston, MA, 1989), pp. 1–16.

[28] J. F. Reynolds, J. Appl. Probab. **9**, 214 (1972).
[29] C. D. Lai, Stochastic Proc. Appl. **7**, 65 (1978).
[30] G. M. Jenkins and D. G. Watts, *Spectral Analysis and its Applications* (Holden-Day, San Francisco, CA, 1968).
[31] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed. (The Clarendon Press/Oxford University Press, New York, 1979), Chap. 1.6, pp. xvi and 426.
[32] F. Spitzer, *Principles of Random Walk* (Springer-Verlag, New York, 2001), Chap. V.
[33] E. Milotti, Phys. Rev. E **51**, 3087 (1995).
[34] S. Erland, Technical Report, Norwegian University of Science and Technology, Trondheim, Norway, 2003 (unpublished).
[35] W. Feller, *An Introduction to Probability Theory and its Applications*, 2nd ed. (Wiley, New York, 1971), Vol. II.
[36] Z. Siwy and A. Fuliński, Phys. Rev. Lett. **89**, 158101 (2002).